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Educing the volume out of the phase space boundary

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Abstract

We explicitly show that, in a system with T-duality symmetry, the configuration space volume degrees of freedom may hide on the surface boundary of the region of accessible states with energy lower than a fixed value. This means that, when taking the decompactification limit (big volume limit), a number of accessible states proportional to the volume is recovered even if no volume dependence appears when energy is high enough. All this behavior is contained in the exact way of computing sums by making integrals. We will also show how the decompactification limit for the gas of strings can be defined from a microcanonical description at finite volume.

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1 Introduction

The density of states or its integrated version measuring the number of accessible sates for a single object (we will denote it as Γ and reserve Ω for the density of states) can be considered as the building block to get a description of a system with a big number N (of the order of Avogadro's number) of such objects. It is precisely this quantity which provides us with the criterion to know whether classical Maxwell-Boltzmann (MB) counting can be applied to a system by inspecting whether the number of accessible states for one object is much bigger than the number of objects in the system.

With the density of states for a single object, it is immediate to compute $q = \sum_{i(\text{states})} \exp(-\beta\epsilon_i)$ by converting the sum into one over energy levels. With q, the partition function for the system with N objects can be calculated by using quantum or classical (MB) statistics.

To compute $\Gamma(E, V, 1)$, where we explicitly note that the number of objects is one, it is necessary to previously look microscopically at the problem of the single object at hand in some configuration in which a finite volume can be defined. In general, this volume will be big as compared with some other length or with energy if we have a fundamental constant with dimensions of length at hand. Traditionally, dealing with objects that can be abstracted as point particles, this is achieved by studying the quantum mechanical dynamics of the object in a box of volume $V = L^d$, where the walls of the box are represented by infinite barriers of potential. However our problem is one in which the generic single object is a string. We take the string as fundamental in various senses, but particularly in a sense in which everything is made of or comes from strings. There is an immediate definition of volume if we take our string and put it by hand in a space which is compact ab initio, for example, if the space is a hypertorus of volume $V = (2\pi R)^d$ where R is the value of the radius for every spatial dimension as a simple election. Volume is then defined as a property of the target space in which the string propagates. This is the kind of box we are using here.

Strings are extended objects. As such, they have some kind of non local features. What happens to them in compact spaces when holes are present is that winding is possible. Winding allows strings to know dimensions as a whole. But, quantum mechanically, the wave function has also a kind of non-local character that allows even a free particle to know that there is a circular dimension and there is also a winding here measuring the number of times the center of mass (one point) winds around the circle. The string

can simultaneously wind in the two fashions so that we cannot distinguish one kind of winding from the other and finally we cannot discern a big circle from a small one because each type of winding respectively contributes to the energy in a way which is directly and inversely proportional to the radius.

Historically, the computations of $\Omega(E, V, 1)$ have been indirect by using canonical (really macrocanonical with chemical potential $\mu = 0$) calculations to get this quantity from the partition function for a single string by inverse Laplace transformation on the inverse temperature variable. Final results have been gotten as an asymptotic approximation valid for high energy (cf., for instance, [1] and [2]). To compute $\Omega(E, V, 1)$ one has to sum over winding and momentum number, when the approximation of substituting the sum over them by a multiple integral holds, the output is a series over the energy independent of the volume. From the sum itself it is also easy to see that there is a regime which is proportional to the volume. Then, two important questions emerge: for the single density of states, can we connect the high energy regime independent of the volume and the one which is proportional to it? For the gas of many strings, is there a well defined relation among energy and volume for which one can say that the system behaves as the gas of strings in uncompactified space? We will see that these two questions are actually entangled and closely related to the way in which the system passes from the low energy regime to the high energy situation as explained in [3].

The starting point of the analysis will be the simple form of Γ as a counting of states with energy lower than a prescribed generic value. $\Gamma(E, R, 1)$ will be computed in section 2. There, we will explain the easy way in which the decompactification limit can be defined and obtained from a simple relation between R and E and how it can be connected to the description of the well known particle gas. We will also explain the difficulties of applying an analogous reasoning to try to get the dependence on R (or the volume) from the sum over windings and momenta when there is no such simple relation between energy and volume. Furthermore, we will also justify the need to quantitatively know more about the approximation of sums by integrals thorough the use of the Euler-Maclaurin (EM) formula. So, section 3 we will be devoted to explain what is relevant about that celebrated formula. In section 4, there appear applications with special emphasis in showing up the way the volume independent situation can be corrected thanks to the EM formula in order to finally get the regime in which $\Gamma(E, R, 1)$ is proportional to the volume. In section 5, we will present and explain the definition of the decompactification regime for the gas with an Avogadro's number of strings

clearly showing that a decompactification limit can be defined upon the microcanonical description of the system. Finally, section 6 will present some comments and possible future prospects for the subject.

2 Exact forms of Γ

Our departure point will be the formulae giving the characteristic perturbative spectrum of a closed string in a toroidal target space, namely

$$\alpha' E^2 = \frac{R^2}{\alpha'}\vec{m}^{\,2} + \frac{\alpha'}{R^2}\vec{n}^{\,2} + 2N + 2\tilde{N} \tag{2.1}$$

$$N - \tilde{N} + \vec{m}\cdot\vec{n} = 0 \tag{2.2}$$

As usual, $\tilde{N}$ and N stand for the right and left moving oscillator numbers and start from -1 if we deal with the closed bosonic string and from zero for the closed superstring. So $\vec{m}$ and $\vec{n}$ are twenty five dimensional integer vectors in the bosonic case and nine dimensional integer ones when supersymmetry is present. The second equation is the left-right level matching condition, which now, in the compact case, whenever finite winding and momentum are present, allows different right and left moving oscillator numbers.

With this, it is easy to write $\Gamma(E,R,1)$ which represents the number of states accessible to one string with energy lower than E. In a hypertorus with d (the number of spatial dimensions, i.e. $d = 9\,\text{or}\,25$) circles with a common radius R related to the volume by $R = V^{\,1/d}/(2\pi)$, Γ reads

$$\sum_{N,\tilde{N}} a(N,\tilde{N}) \sum_{\vec{n},\vec{m}} \delta_{N,\,\tilde{N}-\vec{m}\cdot\vec{n}}\; \vartheta\left(E^2 - \frac{R^2}{\alpha'^2}\vec{m}^{\,2} - \frac{1}{R^2}\vec{n}^{\,2} - \frac{1}{\alpha'}\left(2N + 2\tilde{N}\right)\right) \tag{2.3}$$

Here $a(N,\tilde{N}) = b_N\, b_{\tilde{N}}$ where the b_N are well known natural numbers. They represent the exponentially increasing degeneracy of the oscillations of the string and are the origin of the Hagedorn temperature in the canonical single string partition function $q(\beta)$. $\vartheta(x)$ is the Heaviside function whose value is 1 if $x > 0$, zero if $x < 0$ and $1/2$ when $x = 0$. As expressed in (2.3), it is easy to see that when R is big, $\Gamma(E,R,1)$ converts into a multiple sum in which only the $\vec{m} = \vec{0}$ term (null winding) has a non vanishing contribution because the Heaviside function in (2.3) vanishes (for $\vec{m} \neq \vec{0}$) when $E < R/\alpha'$.

So, we have

$$\Gamma(E,R,1)=\sum_N b_N^2\sum_{\vec{n}}\ \vartheta\left(E^2-\frac{1}{R^2}\vec{n}^{\,2}-\frac{1}{\alpha'}4N\right)\quad(E<R/\alpha')\qquad(2.4)$$

This can be seen as Γ computed for an infinite number of relativistic particles with squared masses $4N/\alpha'$. There is a total polarization degeneracy of value b_N^2 and the momenta are fully discrete. It is also noticeable that the left-right level matching condition (a fully closed-stringy ingredient) has become as trivial as in the open space case. Anyway there still remains a stringy ingredient in the collection of particles as a whole that is reflected in the exponential growth of the coefficients b_N. This is the string seen as a collection of quantum fields or analog model.

Now it is direct to use the standard argument to compute the sum as an integral based on the fact that counting the number of accessible states corresponds geometrically to compute the number of points of a cubic lattice inside a sphere defined by the argument of the Heaviside function in (2.4). This is just accomplished by integration. This counting approximates well the sum as long as the radius of the sphere is big enough to contain a big number of points so as to minimize the relative error that appears because of the clear bad counting near the spherical surface. We will see in section 4 a precise way of computing this error. The result of counting points by making an integral will finally give a term proportional to the volume V.

It seems common sense to apply the same reasoning to the crude $\Gamma(E,R,1)$. However, there appear difficulties. Phase space geometry is slightly more complex but, at first sight, tractable as the volume enclosed by a kind of an elliptical deformation of an eighteen-dimensional sphere (to be concrete, we focus on the superstring) producing two groups of nine semi-axes each of squared length $(E^2-\frac{1}{\alpha'}(2N+2\tilde{N}))R^2$ and the T-duality transformed (i.e, what is obtained under $R\longrightarrow\alpha'/R$) respectively. What spoils the reasoning of counting states by computing volumes by integration is the closed string level matching condition that makes the lattice one that cannot be built upon replicating a cubic cell. However, with respect to the problem of knowing how the volume dependence works in relation with computing integrals, the fictitious system obtained by putting by hand $N=\tilde{N}$ instead of the constraint in (2.2) will serve. The reasoning on counting states by integrating to get a volume is then immediate and we may try to forget for a moment what the error linked to surface effects could be, although we anticipate it

will be crucial. It is clearly easy to realize that the phase-space volume of the elliptical object with the given axes is independent of R because, just with a simple change of variables with unit Jacobian, we make R disappear from the argument of the Heaviside function. So, surprisingly enough, when the integral is a good approximation to the sum, no volume dependence appears. Now, if we put this result and the computation of the big volume limit (which gives the same result for the closed string with constraint and the fictitious trivialized system) together, it is clear that the difference between the sum and the integral, i.e. what could be called the rest from a point of view approximating sums by integrals, must be where volume dependence lies. It is then mandatory to study the exact relation between computing sums by means of integrals and sums themselves. This will be equivalent for us to studying the Euler-Maclaurin summation formula in a multivariate form.

3 The Euler-Maclaurin formula to make sums by means of integrals

Euler-Maclaurin summation formula was conceived and has mainly been used as a celebrated formula for numerical integration by converting integrals into sums. This formula was discovered independently by Euler and Maclaurin around 1740. It can immediately be seen to be an extension of the trapezoidal rule for integral approximation of one single variable functions. We need it here to make the converse interpretation; i.e., to compute sums by calculating integrals. Consequently, we need a multivariate generalization.

Even having found a reference to the existence of such a multivariate generalization in [4], we have not been able to find the equation giving it. So, in the appendix, we present a generalization based on a systematic application of the Gauss theorem to a natural extension of the beautiful proof for this formula presented in [5](see also [6] and the appendix here) based on the periodic Bernoulli functions of one single variable. The main idea roots on looking at the problem in a distributional sense[2] and using the periodic Dirac delta function as the natural mediator to convert sums into integrals and vice versa. The Euler-Maclaurin summation formula will emerge after realizing that the periodic Dirac delta function in n-dimensions will produce

[2]Although it can also be understood only with the need of the Stieltjes integration as in [6].

one of the distributional generalizations of the Bernoulli periodic functions. Being more explicit,

$$\delta_{\mathrm{per}}(\vec{x}) = \sum_{\vec{k}\in\mathbb{Z}^n} \delta(\vec{x}-\vec{k}) \tag{3.1}$$

in such way that

$$\begin{aligned}\int_{\mathbb{R}^n} \mathrm{d}\vec{x}\, g(\vec{x})\ \vartheta(f(\vec{x}))\,\delta_{per}(\vec{x}) &= \sum_{\vec{k}\in\mathbb{Z}^n} \int_{\mathbb{R}^n} \mathrm{d}\vec{x}\, g(\vec{x})\ \vartheta(f(\vec{x}))\,\delta(\vec{x}-\vec{k}) \\ &= \sum_{\vec{k}\in D}{}' g(\vec{k}) = \sum_{\vec{k}\in\mathbb{Z}^n} g(\vec{k})\,\vartheta\left(f(\vec{k})\right)\end{aligned} \tag{3.2}$$

where $\sum'$ is a summation modified by taking only half of $g(\vec{k})$ when $\vec{k}\in\partial D$, the surface defined by the equation $f(\vec{x})=0$ which bounds the compact region D as a subset of $\mathbb{R}^n$. The one half factor comes from the definition of the Heaviside function of one variable given above. It is worth to stress that, when a product of more than one Heaviside function is necessary to define the domain D, the prime will mean a certain combination of one half terms for lattice points on the boundary (see the appendix). Now things start to produce a multivariate Euler-Maclaurin formula because

$$1 - B_0(\vec{x}) = \delta_{\mathrm{per}}(\vec{x}) \tag{3.3}$$

Here $B_0(\vec{x})$ is obtained from $\vec{B}_1(\vec{x})$ from the relation

$$B_0(\vec{x}) = \vec{\nabla}\cdot\vec{B}_1(\vec{x}) \tag{3.4}$$

where

$$\vec{B}_1(\vec{x}) = -\frac{1}{2\pi i}\sum_{\vec{k}\in(\mathbb{Z}^n)^*} \frac{e^{2\pi i\vec{k}\cdot\vec{x}}}{\vec{k}^{\,2}}\,\vec{k} = -\frac{1}{2\,\pi}\sum_{\vec{k}\in(\mathbb{Z}^n)^*} \frac{\sin(2\pi\vec{k}\cdot\vec{x})}{\vec{k}^{\,2}}\,\vec{k} \tag{3.5}$$

which is an expression of $\vec{B}_1(\vec{x})$ thorough its Fourier expansion that emphasizes its real character.

An exact formula can now be written in which it is explicit how a multiple sum can be computed thorough integrals, to wit

$$\sum_{\vec{k}\in D}{}' \; g(\vec{k}) = \int\limits_{D} \mathrm{d}\vec{x}\; g(\vec{x}) - \int\limits_{\partial D} \mathrm{d}\vec{S}\cdot\left(g\left(\vec{x}\right)\vec{B}_1\right) + \int\limits_{D} \mathrm{d}\vec{x}\left(\vec{\nabla}g\right)\cdot\vec{B}_1 \qquad (3.6)$$

Where we have used that

$$\int\limits_{\mathbb{R}^n} \mathrm{d}\vec{x}\,\vec{B}_1\cdot\left(\vec{\nabla}f\right)\delta\left(f\left(\vec{x}\right)\right)\; g\left(\vec{x}\right) = -\int\limits_{\partial D} d\vec{S}\cdot\left(g\left(\vec{x}\right)\vec{B}_1\right) \qquad (3.7)$$

with ∂D the boundary of the region enclosed by the surface defined by $f\left(\vec{x}\right) = 0$ with a unit vector normal to the surface given by $\vec{e}_{\mathbf{s}} = -(\vec{\nabla}f)/|\vec{\nabla}f|$. By simply taking $\vec{B}_1 = -\vec{\nabla}f/|\vec{\nabla}\,f|$ and $g(\vec{x}) = 1$, (3.7) is also useful to get $|\mathrm{d}\vec{S}|$ when the volume element is known because it is valid for any $\vec{B}_1$ and $g(\vec{x})$.

In the appendix we will show how to obtain more terms by using the other multivariate generalizations of the Bernoulli functions given by $B_{n\,(\mathrm{even})}$ and $\vec{B}_{n\,(\mathrm{odd})}$.[3]

4 Simple applications to get $\Gamma(E, R, 1)$ for particles and strings

It seems justified to start this section satisfying our curiosity by the simple application of Euler-Maclaurin's formula to the relativistic classical gas. We suppose we have a single particle (of mass M) with accessible energy levels $E^2 = M^2 + \vec{n}^{\,2}/R^2$ in d spatial dimensions. The lattice of momenta is the one generated by the integers, i.e. $\mathbb{Z}^d$. The equation defining the domain D can be taken as $f(\vec{x}) = (E^2 - M^2)R^2 - x^2 = 0$ where $x = |\vec{x}|$. It gives a sphere in phase space of radius $\rho = R(E^2 - M^2)^{1/2}$. The vector $\vec{e}_{\mathbf{s}}$ is the

[3] If, after several applications of the Gauss theorem, the integral over D representing the rest is suppressed, one finds that the remaining terms generate an asymptotic series.

unit radial vector. With all this we have

$$\sum_{\vec{n}} \vartheta(E^2 - M^2 - \vec{n}^{\,2}/R^2) = \frac{2\pi^{d/2}}{d\,\Gamma(d/2)}\,\rho^d + \rho^{d/2} \sum_{\vec{k}\neq\vec{0}} k^{-d/2}\,\mathrm{J}_{d/2}\,(2\,\pi\,k\,\rho)$$
$$\xrightarrow{\rho\longrightarrow+\infty} \frac{2\pi^{d/2}}{d\,\Gamma(d/2)}\,\rho^d + \frac{1}{\pi}\,\rho^{(d-1)/2} \sum_{\vec{k}\neq\vec{0}} k^{-(d+1)/2}\,\cos\left(2\,\pi\,k\,\rho - \left(\frac{\pi(d+1)}{4}\right)\right) \tag{4.1}$$

We see that, as expected, the absolute error coming from the surface term grows with the radius of the sphere, i.e. grows with energy and R, but the relative error of the surface contribution to the phase space volume of the sphere decreases with energy and R. In terms of the configuration space volume V, we see that, when ρ is big, the relative error goes as $V^{-(d+1)/(2d)} \xrightarrow{\text{big } d} 1/\sqrt{V}$. We remark that the phase space surface term (what would be the error when approximating sums by integrals) oscillates changing its sign. It is important to stress that, for a single object with several discrete mass levels, whenever a mass channel opens, there is a relevant contribution from the surface corrections because $\rho < 1$ around that mass. The contribution just vanishes when the channel opens.

The next application will be to the computation of the number of accessible states for a single string for the fictitious trivialized system of section 2 with $N = \tilde{N}$ and unrestricted winding and momentum vectors. In this case $g(\vec{x}) = 1$ and consequently $\vec{\nabla} g = \vec{0}$. The domain D is the region bounded by the surface given by

$$1 = \frac{\vec{m}^{\,2}}{\frac{\alpha'^{\,2}}{R^2}\,(E^2 - 4N/\alpha')} + \frac{\vec{n}^{\,2}}{R^2\,(E^2 - 4N/\alpha')} \tag{4.2}$$

Here we keep N fixed because we are interested in studying the dependence on the volume V by summing first on windings and momenta. Since $g(\vec{x})$ is constant, the multivariate Euler-Maclaurin formula just gives the difference between the sum and the integral in terms of the first Bernoulli function as

$$\sum_{(\vec{m}\,,\,\vec{n})\in D}{}' 1 - \int_D \mathrm{d}\vec{m}\,\mathrm{d}\vec{n} = -\int_{\partial D} \mathrm{d}\vec{S}\cdot\vec{B}_1 \tag{4.3}$$

The integral over D is the volume of the ellipsoidal region in the 18-dimensional phase space of momenta and windings that, as stressed in the introduction, is clearly independent of the radius of compactification.

Our challenge will be to show (rather to prove out) that, when taking the limit of big volume, the surface term, that gives the error made when substituting the sum by the integral, is able to produce a contribution that cancels the integral approximation and give also the decompactification limit term that goes as V. We will take $d = 9$ and focus on the superstring case. After all, the bosonic string can only be used in this context as a sort of a toy model.

To write the surface integral, we take polar spherical coordinates in both the nine dimensional space of the variable $\vec{m}$ and the one for the variable $\vec{n}$. Next, we take $n = |\vec{n}| = Rr\,\cos\theta;\ m = |\vec{m}| = r(\alpha'/R)\,\sin\theta$ with $\theta \in [0, \pi/2]$. This turns (4.2) for the ellipsoidal surface into the equation for a sphere of radius $(E^2 - 4N/\alpha')^{1/2}$. The surface element is given by

$$\frac{\mathrm{d}\,\vec{S}}{\mathrm{d}\theta\ \mathrm{d}\Omega_{\mathbf{m}}\ \mathrm{d}\Omega_{\mathbf{n}}} = \alpha'^{\,9}\left(E^2 - \frac{4N}{\alpha'}\right)^{17/2}\ \sin^8\theta\,\cos^8\theta\sqrt{\frac{R^2}{\alpha'^{\,2}}\,\sin^2\theta + \frac{\cos^2\theta}{R^2}}\ \vec{e}_{\mathbf{s}},$$

where $\Omega_{\mathbf{n},\,\mathbf{m}}$ are the solid angles in the momentum and winding variables respectively. The normal to the surface unit vector is given by

$$\vec{e}_{\mathbf{s}} = \left(\frac{R^2}{\alpha'^{\,2}}\,\sin^2\theta + \frac{\cos^2\theta}{R^2}\right)^{-1/2}\ \left(\vec{e}_{\mathbf{m}}\,\frac{R}{\alpha'}\sin\theta\,,\ \vec{e}_{\mathbf{n}}\frac{\cos\theta}{R}\right)$$

with $\vec{e}_{\mathbf{m},\,\mathbf{n}}$ the unit radial vectors for the space of the continuous momentum and winding variables respectively.

We use (3.5) to get $\int_{\partial D}\mathrm{d}S\ \vec{k}\cdot\vec{e}_{\mathbf{s}}\,\sin\left(2\pi\vec{k}\cdot\vec{x}\right)/\vec{k}^{\,2}$ for a fixed mode $\vec{k} = (\vec{k}_{\mathbf{m}}\,,\,\vec{k}_{\mathbf{n}})$ and, after that, sum over $\vec{k} \neq \vec{0}$. The surface integral equals

$$\begin{aligned}\frac{1}{2\,\pi}\alpha'^{\,9}\rho^{17}\sum_{(\vec{k}_{\mathbf{m}}\,,\,\vec{k}_{\mathbf{n}})\neq\vec{0}}\int \mathrm{d}\theta\,d\Omega^{(8)}_{\mathbf{m}}\,\mathrm{d}\Omega^{(8)}_{\mathbf{n}}\,\mathrm{d}\theta_{\mathbf{m}\,,\mathbf{7}}\,\mathrm{d}\theta_{\mathbf{n}\,,7}\ \left(\sin\theta_{\mathbf{m}\,,7}\sin\theta_{\mathbf{n}\,,7}\right)^7\\ \times\ \left(\sin\theta\,\cos\theta\right)^8\ \frac{k_{\mathbf{m}}\frac{R}{\alpha'}\sin\theta\cos\theta_{\mathbf{m}\,,7} + k_{\mathbf{n}}\frac{1}{R}\cos\theta\cos\theta_{\mathbf{n}\,,7}}{k^2_{\mathbf{m}} + k^2_{\mathbf{n}}}\\ \times\ \sin\left(2\pi\rho\left(k_{\mathbf{m}}\frac{\alpha'}{R}\sin\theta\cos\theta_{\mathbf{m}\,,7} + k_{\mathbf{n}}\,R\cos\theta\cos\theta_{\mathbf{n}\,,7}\right)\right)\end{aligned} \tag{4.4}$$

where $\rho = (E^2 - 4N/\alpha')^{1/2}$, and $\Omega^{(8)}_{\mathbf{n},\mathbf{m}}$ are the solid angles for the eight-dimensional subspaces of the winding and momentum variables orthogonal to the $X_{\mathbf{m},9}$ and $X_{\mathbf{n},9}$ axes respectively. We have chosen the fixed vector $\vec{k}$ along a fixed direction taking $\vec{k}_{\mathbf{m}}$ along the $X_{\mathbf{m},9}$ axis and $\vec{k}_{\mathbf{n}}$ along $X_{\mathbf{n},9}$.

If we expand the sine function in the last factor of (4.4) as $\sin(2\pi\rho k_{\mathbf{m}}\frac{\alpha'}{R}...)\cos(2\pi\rho k_{\mathbf{n}} R...) + \cos(..)\sin(...)$ we see that when $k_{\mathbf{n}} \neq 0$ and $\rho\alpha'/R \ll 1$ the sinus function of the $1/R$ term varies slowly as long as the cosine does rapidly producing a small contribution to the integral. So, finally, in this limit (formally, $R \longrightarrow +\infty$) only the $k_{\mathbf{n}} = 0$ contributes significantly and we can write for the integral

$$\frac{R}{2\,\pi}\alpha'^{\,8}\rho^{17}\sum_{\vec{k}_{\mathbf{m}}\neq\vec{0}}\int \mathrm{d}\theta\,\mathrm{d}\Omega^{(8)}_{\mathbf{m}}\,\mathrm{d}\Omega^{(8)}_{\mathbf{n}}\,\mathrm{d}\theta_{\mathbf{m},7}\,\mathrm{d}\theta_{\mathbf{n},7}\,(\sin\theta_{\mathbf{m},7}\sin\theta_{\mathbf{n},7})^{\,7}\,\sin^9\theta \times\,\cos^8\theta\,\frac{\cos\theta_{\mathbf{m},7}}{k_{\mathbf{m}}}\sin\left(2\pi\rho\,k_{\mathbf{m}}\frac{\alpha'}{R}\sin\theta\cos\theta_{\mathbf{m},7}\right) \tag{4.5}$$

The next step is to realize that whenever $\rho\alpha'/R \ll 1$ the sum over $\vec{k}_{\mathbf{m}} \neq \vec{0}$ can be substituted by an integral. Since the sum does not include the nine dimensional zero vector, we should exclude the region in the lattice around the origin bounding it by a cube centered in the origin. It is worth to note that one must be careful with the points of the lattice that belong to the boundary of the cube because their contribution will be corrected by powers of one half factors coming by the vanishing arguments of the Heaviside functions defining the domain inside the whole space. Schematically what we have is

$$\sum_{\vec{k}_{\mathbf{m}}\neq\vec{0}} = \int_{(\mathbb{R}^{\,9}-\text{cube})} \quad + \quad \text{corrections for points that belong to the boundary} \tag{4.6}$$

In other words, the integral would approximate a sum of the primed type previously described. To compute a sum without any restriction over the points just on the boundary, we must correct the integral with the adequate contribution for these points (see the appendix). The integral taking out the cube can be written as the difference between the integral to the whole space minus the integral over the cubic region. This trick allows us to take advantage of the spherical coordinates to compute for the whole space and the Cartesian coordinates when calculating over the cube.

By making the sum over $\vec{k}_{\mathbf{m}}$ as an integral including $\vec{0}$ just after the other integrals are performed, we get that (4.5) equals the volume of a nine dimensional sphere of radius $R\,\rho$, i.e. $2\pi^{9/2}(R\,\rho)^9/(9\Gamma(9/2))$. Now one has to compute an integral over the region bounded by the cubic surface centered in the origin. In this compact region, $|\vec{k}_{\mathbf{m}}| \leq \sqrt{9}$. This together with the fact that R is big makes $\sin(2\pi\rho\, k_{\mathbf{m}}\,\alpha' \sin\theta\cos\theta_{\mathbf{m},7}/R) \simeq 2\pi\rho\, k_{\mathbf{m}}\,\alpha' \sin\theta\cos\theta_{\mathbf{m},7}/R$ a good approximation as long as R formally goes to infinity.

The integral over the cubic region can then easily be done using Cartesian coordinates. We have

$$\alpha'^{\,9}\rho^{18}\int_{-1}^{+1} \mathrm{d}k_{\mathbf{m},1}...\int_{-1}^{+1} \mathrm{d}k_{\mathbf{m},9}\int \mathrm{d}\theta\,\mathrm{d}\Omega_{\mathbf{m}}^{(8)}\,\mathrm{d}\Omega_{\mathbf{n}}^{(8)}\,\mathrm{d}\theta_{\mathbf{m},\mathbf{7}}\,\mathrm{d}\theta_{\mathbf{n},7}\,\left(\sin\theta_{\mathbf{m},7}\sin\theta_{\mathbf{n},7}\right)^7$$
$$\times\ \sin^{10}\theta\cos^8\theta\ \cos^2\theta_{\mathbf{m},7}\ =\ \frac{2\pi^{\,9}\rho^{18}}{18\,\Gamma(9)}\,2^{\,9}\,\alpha'^{\,9} \tag{4.7}$$

The presence of α' means that this term is of stringy nature. The factor $2^{\,9}$ is the only contribution coming for the integrals partially representing the sum over $\vec{k}_{\mathbf{m}}$. Now we need to correct this result by adding the contribution of the points on the boundary. The rule is clear (see the appendix): for the lattice vectors with a single component equal to ±1 we have to add a term $1/2(f(1,0,...,0)+f(-1,0,...,0)+\text{permutations})$, for the lattice vectors on the boundary which have two components equal to ±1 we have to add other term $1/2^{\,2}(f(1,1,0,...,0)+f(-1,1,0,...,0)+f(-1,-1,0,...,0)+\text{permutations})$ and so on. The function f is in fact a constant because inside and over the cube the sinus function can be approximated by its argument when R is big enough. For each vector we then have a constant contribution given by $2\pi^{\,9}\,\alpha'^{\,9}\rho^{18}/(18\,\Gamma(9))$. The combinatorics involved is not too difficult. The vectors that contribute $1/2$ of the constant value are the ones with null components but one single $+1$ or -1; there are $2\binom{9}{1}$ such vectors. The ones which give $1/2^{\,2}$ of the constant value are the ones with two components equal to ±1; there are $2^{\,2}\binom{9}{2}$ of them, and so on. The total correction from

all the lattice points on the cubic boundary is

$$\frac{2\pi^{\,9}\rho^{18}}{18\Gamma(9)}\;\alpha'^{\,9}\sum_{i=1}^{9}\binom{9}{i}=\frac{2\pi^{\,9}\rho^{18}}{18\,\Gamma(9)}\;\alpha'^{\,9}\;\left(2^{\,9}-1\right) \tag{4.8}$$

Consequently, we have shown the way the volume reappears if one starts from a regime in which the integral is a good approximation to the sum to compute the number of accessible states for a single object. The whole calculus for the trivialized system can be summarized as

$$\begin{aligned}&\sum_{\vec{n},\vec{m}}\vartheta\left(E^{\,2}-\frac{R^{\,2}}{\alpha'^{\,2}}\vec{m}^{\,2}-\frac{1}{R^{\,2}}\vec{n}^{\,2}-\frac{4\,N}{\alpha'}\right)=\\&\qquad\int\limits_{\mathbb{R}^9\times\mathbb{R}^9}d\vec{m}\,\mathrm{d}\vec{n}\;\;\vartheta\left(E^{\,2}-\frac{R^{\,2}}{\alpha'^{\,2}}\vec{m}^{\,2}-\frac{1}{R^{\,2}}\vec{n}^{\,2}-\frac{4\,N}{\alpha'}\right)-\int\limits_{\text{Ellip.}}\mathrm{d}\vec{S}\cdot\vec{B}_1\\&\qquad\xrightarrow{\rho\alpha'/R\ll 1}\;\;\frac{\alpha'^{\,9}\rho^{18}\,\pi^{\,9}}{362880}+\left[\frac{32\,\pi^{4}\;(\rho\,R)^{\,9}}{945}-\frac{\alpha'^{\,9}\rho^{18}\,\pi^{\,9}}{362880}\left(2^{\,9}-\left(2^{\,9}-1\right)\right)\right]\end{aligned} \tag{4.9}$$

We are now prepared to confront the pure string system. The complication added to the trivialized system is the left-right level matching condition that is a function of winding and momentum numbers. We have to find a representation of the Kronecker delta in (2.2) in a way that can be used by the Euler-Maclaurin formula thorough a function of real variable as $g(\vec{x})$ in (3.2). To this purpose, we represent the Kronecker delta as an integral

$$\int\limits_{-1/2}^{+1/2}\mathrm{d}t\;\;\exp\left(-2\,\pi\,\mathrm{i}\,t\,(\Delta-\vec{m}\cdot\vec{n})\right)=\delta_{\Delta\,,\,\vec{m}\cdot\vec{n}}=\frac{\sin(\pi\,(\Delta-\vec{m}\cdot\vec{n}))}{\pi\,(\Delta-\vec{m}\cdot\vec{n})} \tag{4.10}$$

which holds for $\Delta=\tilde{N}-N$ integer and $\vec{m}$, $\vec{n}$ integer vectors.

The number of accessible states for one string which has energy lower

than a fixed value E can then be written as

$$\Gamma(E,R,1) = \sum_{N,\,\tilde{N}} a(N,\tilde{N}) \sum_{\vec{n},\vec{m}} \int_{-1/2}^{+1/2} \mathrm{d}t \; \mathrm{e}^{-2\,\pi\,\mathrm{i}\,t\,(\Delta - \vec{m}\cdot\vec{n})} \times \vartheta\left(E^2 - \frac{R^2}{\alpha'^2}\vec{m}^{\,2} - \frac{1}{R^2}\vec{n}^{\,2} - \frac{1}{\alpha'}\,(4N + 2\Delta)\right) \tag{4.11}$$

For given oscillator numbers, we can represent the sums over windings and momenta as integrals by using the multivariate Euler-Maclaurin formula to get

$$\begin{aligned}
&\int_{-1/2}^{+1/2} \mathrm{d}t\, \mathrm{e}^{-2\,\pi\,\mathrm{i}\,t\,\Delta} \sum_{\vec{n},\vec{m}} \mathrm{e}^{2\,\pi\,\mathrm{i}\,t\,\vec{m}\cdot\vec{n}} \\
&\qquad \times \vartheta\left(E^2 - \frac{R^2}{\alpha'^2}\vec{m}^{\,2} - \frac{1}{R^2}\vec{n}^{\,2} - \frac{1}{\alpha'}\,(4N + 2\Delta)\right) \\
&\quad = \alpha'^{\,9}\; E^{\,18} \int_{-1/2}^{+1/2} \mathrm{d}t \quad \mathrm{e}^{\,-2\,\pi\,\mathrm{i}\,t\,\Delta} \left[\int_D \mathrm{d}\vec{m}\,\mathrm{d}\vec{n} \quad \mathrm{e}^{\,2\,\pi\,\mathrm{i}\,t\,\alpha'\,E^{\,2}\vec{m}\cdot\vec{n}} \right. \\
&\qquad - 2 \int_{\mathbb{R}^{\,9}\times\mathbb{R}^{\,9}} \mathrm{d}\vec{m}\,\mathrm{d}\vec{n}\; \mathrm{e}^{\,2\,\pi\,\mathrm{i}\,t\,\alpha'\,E^{\,2}\,\vec{m}\cdot\vec{n}} \left(\frac{R\vec{m}}{\alpha'}\,,\,\frac{\vec{n}}{R}\right)\cdot\vec{B}_1\left(\frac{E\alpha'\vec{m}}{R}\,,\,E\,R\,\vec{n}\right) \\
&\qquad\qquad \times E^{\,-1}\,\delta\left(1 - \vec{m}^{\,2} - \vec{n}^{\,2} - \frac{4N + 2\Delta}{E^{\,2}\,\alpha'}\right) \\
&\qquad \left. + 2\pi\,\mathrm{i}\,t\,E \int_D \mathrm{d}\vec{m}\,\mathrm{d}\vec{n} \quad \mathrm{e}^{\,2\,\pi\,\mathrm{i}\,t\,\alpha'E^{\,2}\,\vec{m}\cdot\vec{n}} \left(R\,\vec{n}\,,\,\frac{\alpha'\vec{m}}{R}\right)\cdot\vec{B}_1\left(\frac{E\alpha'\vec{m}}{R}\,,\,E\,R\,\vec{n}\right) \right]
\end{aligned} \tag{4.12}$$

where the domain D is the region bounded by the surface defined by the vanishing of the argument of the Dirac delta that appears because it seems convenient to represent the surface integral as a volume integral over the whole (continuous) phase space.

Now one has to compute the limit of Γ when the volume gets big to see the way it finally becomes proportional to the volume of the hypertorus as

one departs from a situation in which no volume dependence appears because the first integral is a good approximation. When energy is high enough, one gets that D, the domain of integration[4], is such that the number of points inside the boundary is really big when compared to the error over the surface that increases, but with a lower power of ρ'.

Let us now give the detailed proof of the way the high volume limit shows up from (4.12). The first step is just to expand $\mathrm{e}^{2\pi\,\mathrm{i}\,t\alpha'\,E^2\vec{m}\cdot\vec{n}}$ as a series of powers of its argument. The zero order in this expansion gives, for the first and second integrals, the same contribution as for the fictitious trivialized toy model (after changing ρ by ρ'), so the analysis made for this example applies for it and finds here its real purpose. The contributions from the odd powers vanish[5]. For the last integral in (4.12), it is easy to see that the contributions from the even powers of the expansion of the exponential also vanish.

We have to compute a sum over $\vec{k}_{\mathbf{m}}$ in the integrals containing $\vec{B}_1$. This sum can be computed as an integral over the real variable $\vec{k}$ plus the contribution from taking out the point $\vec{k}_{\mathbf{m}} = \vec{0}$ (see (4.6)). The contribution integrated over $\vec{k} \in \mathbb{R}^9$ coming from the $(2j-1)$-th power in the third integral exactly cancels the part integrated over $\vec{k}$ from the $2j$-th power of the second integral (always in the limit in which $\rho^{'}\alpha'/R \ll 1$).

Finally, for both sums over $\vec{k}_{\mathbf{m}}$, we are left with the contributions from the points on the boundary of the cube centered at the origin plus the integral over this domain (cf. (4.6)). It is easy to see that these contributions from the $2j-1$-th order in the third integral and those from the $2j$-th order in the second one add up to exactly cancel the $2j$-th order of the first integral, which is the part manifestly independent of the configuration volume V. This is the rigorous proof of the connection between the particle gas regime and the volume independent situation for $\Gamma(E, R, 1)$.

[4] D is always an ellipsoid with semi-axes given, after putting $\rho' = (E^2 - (4N + 2\Delta)/\alpha')^{1/2}$, by $\rho'\alpha'/R$ and $\rho' R$

[5] We could have written $\cos 2\pi\, t\,(\Delta - \vec{m}\cdot\vec{n})$ instead of the exponential as the integrand in (4.10) to make this manifest.

5 The limit of decompactification for the gas of strings

Now it is mandatory to try to get the first consequence of what we have learned. We think that the issue on whether the uncompactified regime for the gas of strings can be recovered from the compact description deserves a certain reanalysis after putting together what we know now on $\Gamma(E, R, 1)$ and what is presented in [3].

By using the inverse Laplace transform techniques the best that has been obtained for the density of states of the gas of strings with compact dimensions is a series that converges whenever [2]

$$E > KR^d/\alpha'^{(d+1)/2} \equiv E_H \tag{5.1}$$

(d is the number of big dimensions). The term on the right hand side of this inequality is a sort of Hagedorn characteristic energy. The important point here is that in [3] it is explained that, when R is big enough as compared with $\sqrt{\alpha'}$, K can actually be computed to give a precise physical definition of the Hagedorn energy E_H as the energy at which the system reaches the Hagedorn temperature or approaches it from a low energy regime dominated by a gas of massless objects. Hence, the Hagedorn regime will be defined as the regime at which the massive modes appear in the gas. These modes have masses that are proportional to $1/\sqrt{\alpha'}$ which is the characteristic mass scale of the stringy effects. They are then oscillators, whose mass is a positive integer times $1/\sqrt{\alpha'}$, or windings of mass R/α'. To be more concrete, when one has a system with an Avogadro's number of strings, looking at its Hagedorn regime dominated by oscillators means looking at the system when the the energy is such that $E/N_H > 2/\sqrt{\alpha'}$, where N_H is the number of strings that the gas presents at the Hagedorn energy as given by the characteristic Hagedorn temperature (cf. [3]).

$$E_H = C_{10}\, V\, \beta_H^{-10} \tag{5.2}$$

where C_{10} is a known constant.

The bound $2/\sqrt{\alpha'}$ is precisely the minimum mass of a non vanishing vibrational mode of a single string. At this point, we must take into account that there is a relationship between the number of strings and energy and also volume (or R). It is the result of the fact that our microcanonical description is one very special (it is not rigorously microcanonical) for which

the chemical potential μ vanishes. This condition for equilibrium gives in [3] that $N_H \sim V/\alpha'^{d/2}$, which simply states that the number of strings at this characteristic energy is finally proportional to the volume, like in the black-body in equilibrium with a gas of photons. This is so because one can always think of the system as reaching or approaching the Hagedorn temperature T_H at the energy E_H departing from the low energy regime in which strings are in their massless modes. Thus, E_H is the energy which separates the low energy regime of the gas dominated by the massless modes of the string (the $\alpha' \longrightarrow 0$ limit taken on the mass formula) from the string or high energy regime in which vibrational or winding modes appear. If R is big enough, there is always a low energy regime with massless modes for the gas so we now understand why the condition $E - E_H > 0$ appears the same for different ab initio topologies. This was seen as something notorious in the past.

If one naively thinks that the condition to obtain the uncompactified result is to impose $E < R/\alpha'$, i.e., the replica of the condition over the single string, but now E stands for the total energy, it is easy to see that being in the Hagedorn regime together with this condition forces R to be small instead of big as it was supposed. This way, it would seem that there is no domain of energy and radius for which the characteristic asymptotic behavior in uncompactified space can be recovered. The physical picture of this behavior at the Hagedorn regime is that of a single string absorbing all the energy in its vibrational modes and approaching equilibrium with a sea of low energy strings at the microcanonical Hagedorn temperature. It is precisely the physical picture what makes clear that, to obtain the Hagedorn regime dominated by the vibrational modes characteristic of the uncompactified result, the condition to add to that of being in the Hagedorn regime ($E > E_H$) is $E - E_H < R/\alpha'$. It simply states that the single fat string cannot have winding modes. Now, this is compatible with the fact that $R \gg \sqrt{\alpha'}$. In other words, to get the fat string dominated regime characteristic of having open spatial dimensions, one needs a radius big enough so as to have that the energy of the system minus the Hagedorn energy does not suffice to get windings in the gas in equilibrium (or quasi-equilibrium). Vibrational modes are much lighter and, after reaching E_H, massive strings appear to stay in meta-equilibrium with the sea of massless modes. It is worth to notice that the volume can be very big but the momentum that a string in the gas can have still belongs to a discrete set; we then have and effective decompactification but not a pure one that would be the one in which the nine torus certainly becomes $\mathbb{R}^9$ and the momenta get dense. We will treat

this point later when commenting on the Jeans instability problem.

What happens with the approximate series computed in [2] is that it is adapted to describe the Hagedorn regime when $E - E_H > R/\alpha'$. Mathematically this is analogous to the way in which the integral approximates the sum in Γ to only describe the high energy approximation.

If, from the very beginning, one has that R is of the order of $\sqrt{\alpha'}$, then the mass formula tells us that there is no radiation dominated phase (the standard black-body regime with massless objects) because a massless object with non vanishing momentum is indistinguishable from a massless object with a non vanishing winding. However equation (5.1) is still valid with $d = 0$ giving that high energy means $E > K/\sqrt{\alpha'}$ (cf. the way this condition appears in [2]). In any case, the total energy has to be bigger than the minimum momentum or winding mass which is as big as the mass of a vibrational mode. Of course, if winding modes are present in the gas, from the expression approximating $\Omega(E, R)$, it seems that one cannot get the characteristic negative specific heat phase for the gas with uncompactified dimensions. But, contrary to what is stated in [2] this does not in any way imply that there is no domain in the plane (E, R) for which the uncompactified behavior can be found.

6 Comments and outlook

Before commenting on the contains of the work, may be some more general remarks come into place to justify the interest of the topic we have dealt with. First, it is important to notice that perhaps one of the most active research topics in String Theory has been the one about the Hagedorn phase which is characteristic of a gas of *free* and *perturbatively interacting* strings. Several published works have claimed to have completely solved the problem mainly in two ways: there is really a (may be first order) phase transition at the Hagedorn temperature or the Hagedorn temperature is a maximum one. Things are more subtle than some statements that have been published and are being taken as true. This is the motivation for us to pursue a plan to unambiguously settle what is really well established and what is simply wrong when treating the topic. To this purpose we have already clarified the way quantum statistics effects are taken into account in the problem, the connection between the presence of non extensive terms and negative microcanonical specific heats, and the physical meaning and value of such peculiar

systems [3]. Systems having a negative microcanonical specific heat are a subject of intense active research by the community of theoreticians working on Statistical Mechanics and by atomic and condensed matter experimentalists.

The main criticism that can be made to any work on the free string gas is very well known because one can find it, for example, in a classic review on String Theory as it is the two volume work by M. Green, J. Schwarz and E. Witten [7]. It is a criticism that simply brushes away the Hagedorn problem because Jeans instability and the nucleation of black-holes simply make ill-defined the canonical treatment when interactions are plugged in. It is then often argued, when treating thermodynamical properties of string gases, that no thermodynamic limit can be taken because we are dealing with systems that include gravity, i.e. long range interactions, and no infinite volume (thermodynamic) limit exists (although there is certain controversy on this subject, it seems that the theorem presented in [11] is an exact proof of this fact for classical gravitational interactions). As we are unable to take this limit, the concept of temperature and the idea of a possible phase transition or having a maximum temperature must be discarded because we will be unable to follow the ideas coming from Statistical Physics (see again [7]).

This statement is both wrong and true; it needs clarification. When no thermodynamic limit exists, the canonical description, whose main parameter is temperature (T), becomes ill-defined. The reason is that, from the mechanical point of view, T is a derived parameter, which has no direct significance. On the other hand, a microcanonical description relies upon the use of energy E, a quantity conserved for an isolated system and well defined in phase space. Entropy has also a clear meaning as the logarithm of a volume in phase space (we do not really need a probabilistic definition: $S = k\ln W$ versus $S = \sum_i p_i \ln p_i$). In Statistical Thermodynamics, extensivity is assumed and the thermodynamical limit (as $V \to \infty$) is taken. It is in this situation that both canonical and microcanonical descriptions appear to be equivalent. A phase transition will show as a non homogeneity of the systems, a non extensivity.

In our problem, even when the string coupling vanishes ($g_s = 0$) and we do not have long range interactions, we do not have an extensive entropy [3]. It can also be shown that microcanonical and canonical descriptions are, in some cases, inequivalent. After introducing interactions in our system, no thermodynamic limit can be taken as the result of the anti-screening properties of gravitational forces; we will then be forced to treat the system

microcanonically, this being the ensemble adapted for "small" systems. This would let us also study phase transitions in finite volume systems which, by the way, can also be seen experimentally in "small" systems. A very clear exposition of these facts and its relation to negative specific heats can be found in [8] where, among others, classical self-gravitating systems are specifically treated.

For the particular case of self-gravitating systems, where Jeans instability appears, an important effort has been made to treat the problem. In the work by Gross-Perry-Yaffe [9], where an approach from the quantum field theory point of view was presented, the need for a microcanonical treatment was remarked. Since then, the traditional reasoning to avoid Jeans' instability is to put the system in a finite volume as we have done in our work by using a hypertorus. Then the procedure is to rapidly scale the string coupling constant to zero keeping the canonical energy density constant so as to allow the volume to become large. Alternatively, one can fix the coupling as very small and restrict the volume to be large but finite[6]. It is standard lore that to solve infrared divergences due to long range interactions one can use a finite volume. With finite volume there is no $p^\mu = 0$ momentum for the massless objects. Jeans' instability really appears thorough the computation of a non vanishing contribution for the temporal components of the vacuum polarization tensor with $p^0 = 0$ and $\vec{p} \to \vec{0}$ in the pure decompactification limit in which the momenta become dense, in particular, around zero. To avoid Jeans' instability in the canonical description, one can have a really big radius (volume) as compared to $\sqrt{\alpha'}$, but the momenta must still be discrete. This is the precise meaning of 'big but finite' and the difference between an effective decompactification (big R) and a pure decompactification in which the torus becomes $\mathbb{R}^9$ and the momenta become then dense.

A very remarkable fact is that, with an up to now complete ignorance from the string theorist community, the treatment of self-gravitating systems has recently been a very active research field in Statistical Physics [12]. Although, in these works, gravity is treated classically (as Jeans did), very interesting results appear related with phase transitions for the system stemming from negative specific heats and the need for a short distance cut-off (which, by the way, naturally appears in String Theory and should be provided by it). These results are largely seen to be independent of the way the cut-off is

[6]More concretely one has that $\overline{E} < R^{d-2}/(g_s^2 \alpha'^{(d-1)/2}) \sim R^{d-2}/G_N$ as the condition to avoid Jeans' instability

introduced [13]. For a review see [14].

We believe these works signal the importance of a microcanonical treatment for string gases even in the free case. In this work, we have set aside questions related to Jeans instability because we find it compulsory to make a microcanonical treatment of the problem from first principles in String Theory paying special attention to its relationship with the work in [13] (and references there in). In a completely independent way, interesting work [15] has been made on the constraints that the classical Jeans instability (as it is gotten from a quantum field theory in the canonical description) and black-hole nucleation have upon the possibility of really having a Hagedorn "visible" phase that can actually produce black-holes from highly excited (fat) strings at higher energy. However, no direct calculation in the microcanonical description of the Jeans effect has been done. Knowing as we do the results recently found about the microcanonical description of classical self-gravitating systems, we, by no means, find ourselves justified to simply apply the results about Jeans' instability gotten from a fixed temperature description. May be we are being too much conservative, but there are too many subtleties in the subject that deserve some precaution. Finally, to understand how Jeans instability appears related to the string loop expansion computation of the canonical free energy, the work in [10] includes a explanation of the relationship between the perturbative loop expansion and the appearance on Jeans' instability in the computation in [9].

There can be another criticism about the intrinsic instability of the toric box seen in a cosmological context. Problems like decompactifying dimensions, the running dilaton, etc. that are characteristic of toric zero temperature vacuum solutions have to be posed, at finite temperature, in a cosmological picture where the stress tensor of the string gas enters in the Universe evolution equations like it was done long time ago in [16]. Whether a torus is a good background can then only be seen afterwards although lots of works have been published and are actively being produced in this direction (see, as a single example, [17]). Anyhow our main interest in this work is precisely decompactification, although non-dynamical. Sections 4 and 5 have been devoted to show how the toric box can decompactify and we can reach a situation where, for the string gas, all the dimensions are effectively open.

Let us now focus on what has been the precise subject of this article. By computing $\Gamma(E, R, 1)$ for a string in compact space (or its derivative with respect to the energy) by applying a high energy limit analogous to the one in the regular particle gas, one simply finds a limit in which the sum upon

windings and momenta can be converted to an integral with big accuracy. This integral is easily seen to be independent of the volume. After all, converting sums into integrals is a way of getting a more explicit dependence on the volume than the one seen just in the raw sum. Asking why not to use the crude sum instead of the integral has to do with knowing the volume dependence for the thermodynamical system as it happens in the classical free particle gas. What is dropped in this approximation, the phase space boundary contribution, just contains the dependence over the volume that is not present in the high energy limit. The approximation is only valid when energy gets big and the radius of the hypertorus is relatively kept not too big. When the radius grows and so the relation $\alpha' E/R$ is small, the boundary contribution increases its weight and cannot be dropped, it actually produces a term proportional to the volume as in the open space case as long as R becomes bigger with fixed energy. In other words, studying finite size effects in a string system is a subtle issue because of T-duality.

To better see this, we have shown that what is easily seen true as expressed in a set of variables is also true when things are re-expressed in a different set of variables, whenever all the contributions are taken into account. In the problem at hand, this statement means that the high volume limit so effortless computed over $\Gamma(E, R, 1)$ as written in (2.3) can be recovered when the sum over discrete momenta and windings is represented thorough integrals as though they were continuous variables. Furthermore, as long as we keep using the representation of the periodic delta function in terms of phase space coordinates without introducing more Bernoulli periodic functions, the task of getting the high volume limit is still a simple one because the integral representing Γ will finally get a contribution from only one of the Dirac delta functions in the definition of $\delta_{\rm per}(\vec{n}, \vec{m})$. Namely, in the limit $R \longrightarrow +\infty$ of the product $\vartheta\left(\rho'^{\,2} - \frac{\vec{n}^{\,2}}{R^2} - \frac{R^2\vec{m}^2}{\alpha'}\right)\,\delta_{\rm per}(\vec{n}, \vec{m})$ only the contribution for $\delta(\vec{m})$ survives. It is now obvious how the high energy calculations giving an independent of the volume $\Gamma(E, R, 1)$ are related to substituting the sum over windings and momenta in the computation by a multiple integral. What is now new is the fact that the volume independent regime can be connected in a smooth way with the regime proportional to the volume when the equality between the sum and the integrals expressed thorough the EM formula is taken into account. Furthermore, there might be intermediate regimes as for the volume dependence is concerned. As an example, if there really is a regime proportional to the area of the hypertorus

and it is compatible with a Hagedorn regime in which one single string dominates the system with null chemical potential, then the many-string system would have an entropy with a term proportional to the area of the surface enclosing the volume V.

Going over to the technical side, there are certain ambiguities when representing a sum by integrals. The first one is related to the election of the domain D which contains the lattice points we are summing over. For example, it holds that $\sum_{k\in\mathbb{Z}}\,\vartheta(l+0.8-k)\,\vartheta(k-(n-0.2))\vartheta(l-n)=\sum_n^l\,1=\int_{n-0.2}^{l+0.8}\,\mathrm{d}x-(B_1(l+0.8)-B_1(n-0.2))$ which certainly equals $l-n+1$ (to simplify, $n\,,\,l$ are natural numbers, $\langle l.8\rangle=0.8$) and it also equals $\sum_{k\in[n-0.x\,,\,l+0.y]}\,1$. This shows that there are, in general, more than one integration domain and that issue affects the value of the integral over D as the first approximation to the sum. However, this ambiguity is harmless because it reflects the tolerance from the distance of a point in the lattice of integer components to the nearest neighbor. This aspect is negligible in the calculation of Γ. What seems more relevant at first sight is the ambiguity in representing the Kronecker delta function expressing the closed string constraint. It is immediate to see that the integral $\int_0^1\,\mathrm{d}t\,\cos\,2\pi\,t(\Delta-\vec{m}\cdot\vec{n})=\dfrac{\sin\pi(\Delta-\vec{m}\cdot\vec{n})}{\pi\,(\Delta-\vec{m}\cdot\vec{n})}\,\cos\pi(\Delta-\vec{m}\cdot\vec{n})$ is another representation of $\delta_{\Delta\,,\vec{m}\cdot\vec{n}}$ for $\vec{m}\,,\,\vec{n}$ integer vectors that could be used instead of the one in (4.10). It is manifest that, because of the cosine term and after applying the Euler-Maclaurin summation formula, they are different as real variable functions although coincide for integer arguments. Taking the limit $\alpha'\,E^{\,2}\longrightarrow+\infty$ over the first integral upon D on the right hand side of (4.12) for both elections is a good way of estimating the effect of changing the representation. We get that

$$\lim_{\alpha'\,E^{\,2}\longrightarrow+\infty}\,\alpha'\,E^{\,2}\int\limits_{-1/2}^{+1/2}\,\mathrm{d}t\,\cos\,2\,\pi\,t(\Delta-\alpha'\,E^{\,2}\,\vec{m}\cdot\vec{n})=\delta\left(\frac{\Delta}{\alpha'\,E^{\,2}}-\vec{m}\cdot\vec{n}\right)$$

$$\lim_{\alpha'\,E^{\,2}\longrightarrow+\infty}\,\alpha'\,E^{\,2}\int\limits_{0}^{1}\,\mathrm{d}t\,\cos\,2\,\pi\,t(\Delta-\alpha'\,E^{\,2}\,\vec{m}\cdot\vec{n})=\frac{1}{2}\,\delta\left(\frac{\Delta}{\alpha'\,E^{\,2}}-\vec{m}\cdot\vec{n}\right)$$

where we understand that, when $\Delta\neq 0$, we will have contributions from terms with Δ of the order of $\alpha'\,E^2$. It is clear that after summing up the contributions from the three integrals in (4.12), the result of computing the

sum by making integrals is independent of the way we represent the Kronecker delta.

Note added

After disclosing our work, we have found [18](and references therein) in which several multivariate generalizations of the Euler-Maclaurin formula are presented stressing their usefulness to numerically compute integrals by sums. It seems clear that the restrictions this numerical problem imposes makes this project a more difficult mathematical task than it was ours.

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Appendix

Let us present the main ingredients of the Euler-Maclaurin formula as obtained in [5] (see also [6]).

Bernoulli numbers B_n play an important role in the Euler-Maclaurin formula. Bernoulli numbers are concrete values of the Bernoulli polynomials ($B_n = B_n^*(0)$) that can be defined by the recursive formulas:

$$B_0^*(x) = 1 \tag{A.1}$$

$$B_n^{*'}(x) = n\, B_{n-1}^*(x) \tag{A.2}$$

$$\int_0^1 \mathrm{d}x\, B_n^*(x) = 0 \tag{A.3}$$

with $n \geq 1$

The Bernoulli periodic functions are defined as $B_n(x) = B_n^*(\langle x \rangle)$, with $\langle x \rangle = x - [x] =$ fractional part of x. Now M. Lerma in [5] realizes that the odd numbered Bernoulli numbers are zero except $B_1 = -1/2$ and that for all Bernoulli periodic functions $\int_0^1 \mathrm{d}x\, B_n(x) = 0$ except for $n = 0$ that the integral gives 1. To make things easier, it is adequate to use a modified version of the periodic Bernoulli functions that has sense if B_0 is a distribution instead of a function, but that makes the proof of the Euler-Maclaurin summation formula more natural and, what is important to us, easily generalizable to multiple integrals and sums.

The Bernoulli periodic distributions are defined thorough the recursive relations ($n \geq 1$)

$$B_0(x) = 1 - \delta_{\mathrm{per}}(x) \tag{A.4}$$

$$B_n^{'}(x) = nB_{n-1}(x) \tag{A.5}$$

$$\int_0^1 \mathrm{d}x\, B_n(x) = 0 \tag{A.6}$$

Now $B_0(x)$ also verifies (A.6). The differences with the standard definitions of the Bernoulli functions are in $B_0(x)$ and $B_1(x)$. $B_1(x)$ is now the function

$$B_1(x) = \begin{cases} \langle x \rangle - \frac{1}{2} & \text{if } x \notin \mathbb{Z} \\ 0 & \text{if } x \in \mathbb{Z} \end{cases} \tag{A.7}$$

This can be checked in relation with (A.5) by computing $B_1(x) = \int_0^x \mathrm{d}y\, B_0(y)$. It gives

$$\begin{aligned} B_1(x) &= x - \sum_{k \in \mathbb{Z}} \int_0^x \mathrm{d}y \quad \delta\,(y-k) \\ &= x - \sum_{k \in \mathbb{Z}} \left[\vartheta(x-k) - \vartheta(-k)\right] = \begin{cases} \langle x \rangle - \frac{1}{2} & \text{if } x \notin \mathbb{Z} \\ 0 & \text{if } x \in \mathbb{Z} \end{cases} \end{aligned} \tag{A.8}$$

where it is worth to remark that the Dirac delta has been integrated using $\delta\,(x) = \frac{\mathrm{d}}{\mathrm{d}x}\, \vartheta(x)$ with $\vartheta(x)$ the Heaviside function of value $\frac{1}{2}$ for $x = 0$.

The Fourier series representation of the Bernoulli functions will be exceedingly useful when generalizing Lerma's proof to multivariate sums. We have

$$B_n(x) = -\frac{n!}{(2\pi \mathrm{i})^n} \sum_{\substack{k=-\infty \\ k\neq 0}}^{+\infty} \frac{\mathrm{e}^{\,2\pi \mathrm{i} k x}}{k^{\,n}} \tag{A.9}$$

The proof of the EM summation formula is now very easy and starts with $\int_a^b \mathrm{d}x f(x) - \sum_{a\leq k\leq b}^{'} f(k) = \int_a^b \mathrm{d}x\, B_0(x)$. This is not a tautology because we can use that $B_1^{'}(x) = B_0(x)$ to integrate by parts on the right hand side. This can be done successively by using (A.5). The prime over the sum results again from the fact that $\vartheta(0) = 1/2$. And the summation formula emerges beautifully.

To generalize this to multiple sums is straightforward. We can easily write down the periodic Dirac delta function of n variables (see (3.1)) and define the rest of the multivariate Bernoulli periodic functions thorough their Fourier expansions as two series of scalar and vectorial functions.

$$B_n(\vec{x}) = -\frac{n!}{(2\pi \mathrm{i})^n} \sum_{\substack{\vec{k}\in\mathbb{Z} \\ \vec{k}\neq\vec{0}}} \frac{\mathrm{e}^{\,2\pi \mathrm{i}\vec{k}\cdot\vec{x}}}{|\vec{k}|^n} \quad \text{if } n \in (2\mathbb{Z}) \tag{A.10}$$

$$\vec{B}_n(\vec{x}) = -\frac{n!}{(2\pi \mathrm{i})^n} \sum_{\substack{\vec{k}\in\mathbb{Z} \\ \vec{k}\neq\vec{0}}} \frac{\mathrm{e}^{\,2\pi \mathrm{i}\vec{k}\cdot\vec{x}}}{|\vec{k}|^{n+1}}\, \vec{k} \quad \text{if } n \in (2\mathbb{Z}+1) \tag{A.11}$$

Now $1 - B_0(\vec{x}) = \sum_{\vec{k}} \delta(\vec{x}-\vec{k})$, and the recurrence relations are $\vec{\nabla}\cdot\vec{B}_{2n+1}(\vec{x}) = (2n+1)\, B_{2n}(\vec{x})$ and $\vec{\nabla}\, B_{2n+2}(\vec{x}) = (2n+2)\, \vec{B}_{2n+1}(\vec{x})$ with $n \geq 0$.

The expression we get for a sum over integer vectors belonging to a compact region D defined as a subset of $\mathbb{R}^n$, with $\vec{B}_1(\vec{x})$ and $B_2(\vec{x})$ explicitly shown, is

$$\sum_{\vec{k}\in D}^{'} g(\vec{k}) = \int_D \mathrm{d}\vec{x}\; g(\vec{x}) - \int_{\partial D} \mathrm{d}\vec{S}\cdot\Big(g\,(\vec{x})\,\vec{B}_1\Big) + \frac{1}{2}\int_{\partial D} \mathrm{d}\vec{S}\cdot\Big(B_2\,\vec{\nabla} g\Big) - \frac{1}{2}\int_D \mathrm{d}\vec{x}\, B_2\, \Delta\, g \tag{A.12}$$

The prime means that the lattice points $\vec{k}_0$ that are just over the surface boundary will contribute a factor (different from one in general) times $g(\vec{k}_0)$. In the case the domain D is defined as the region bounded by the surface defined by the subset of $\mathbb{R}^n$ of points $\vec{x}$ such that $f(\vec{x}) = 0$, a single Heaviside function will serve to write $\sum_{\vec{k}\in D}{}' g(\vec{k}) = \sum_{\vec{k}\in\mathbb{Z}^n} g(\vec{k})\,\vartheta(f(\vec{k}))$, so the prime will mean that the integer vectors just on the boundary surface, $\vec{k_0}$, will contribute $g(\vec{k_0})/2$ because the value of the step function is $1/2$ when its argument gets null.

In our applications, we have found a sum in which the domain of integration is one that cannot be defined using a single step function, but a combination of products of several of them. More precisely, it is the sum over all the nine dimensional integer $\vec{k}$-vectors with the exception of the vector $\vec{0}$. We have chosen to take out the origin bounding it by a cube of two length side centered at the origin. The domain D will then be the region outside the cube. To get only a contribution to the sum from points in D, we must insert in the sum a factor

$$\sum_{i_1\neq i_2\neq\ldots\neq i_9}\ \sum_{k=1}^{9}\ \frac{1}{k!}\,\frac{1}{(9-k)!}\,\vartheta\left(x_{i_1}^2-1\right)\,\ldots\,\vartheta\left(x_{i_k}^2-1\right)\,\vartheta(1-x_{i_{k+1}}^2)\,\ldots\,\vartheta\left(1-x_{i_9}^2\right) \tag{A.13}$$

From here it is easy to see that, for example, a vertex of the cube like $(1,1,1,1,1,1,1,1,1)$ would contribute

$$\sum_{k=1}^{9}\ \frac{1}{2^9}\binom{9}{k} = 1-\frac{1}{2^9} \tag{A.14}$$

If one wants to compute the sum without any restriction, then one has to sum $1/2^9$ of $g\left((1,1,1,1,1,1,1,1,1)\right)$ to the integrals giving the primed sum.

In our case, we have that $g(\vec{k})$ is constant so one can readily show that the integral faithfully represents the primed sum because the integral over the surface of the cube vanishes.

References

[1] P. Salomonson and B.-S. Skagerstam, *Nucl. Phys.* **B 268** (1986) 349-361.

M. Laucelli Meana, M. A. R. Osorio and J. Puente Peñalba, *Phys. Lett.* **B 400** (1997) 275-283.

[2] N. Deo, S. Jain and C.-I. Tan, *Phys. Rev.* **D 45** (1992) 3641.

[3] M. A. Cobas, M. A. R. Osorio, M. Suárez, *Phys. Lett.* **B 601** (2004) 99-107 (hep-th/040643).

[4] R. Ranga Rao, *Bull. Amer. Math. Soc.*, **67** (1961), 359-361.

[5] M. A. Lerma, "The Bernoulli Periodic Distributions", unpublished, available at http://www.math.northwestern.edu/˜mlerma/papers.

[6] M. A. Lerma, "The Bernoulli Periodic Functions", unpublished.

[7] M. Green, J. Schwarz, E. Witten, *Superstring Theory* Vols. I, II, Cambridge University Press (1987).

[8] D. H. E. Gross, *Chaos, Solitons and Fractals* **13** (2002), 417-430 (cond-mat/0004268).

[9] D. J. Gross, M. Perry, L. G. Yaffe, *Phys. Rev.* **D25** (1982) 330.

[10] E. Álvarez, T. Ortín, M. A. R. Osorio, *Phys. Rev.* **D43** (1991) 3990.

[11] V. Laliena, *Nucl. Phys.* **B 668** (2003) 403 (astro-ph/0303301).

[12] T. Padmanabhan, *Ap. J. Suppl.* **71** (1989) 651.
P.-H. Chavanis, *Phys. Rev.* **E 65** (2002) 056123.
P.-H. Chavanis, C. Rosier, C. Sire, *Phys. Rev.* **E 66** (2002) 036105.
V. Laliena, *Phys. Rev* **E 59** (1999) 4786.
H. J. de Vega, N. Sánchez, B. Semelin, *Phys. Rev.* **D 63** (2001) 084005.

[13] E. Follana, V. Laliena, *Phys. Rev.* **E 61** (2000) 6270 (cond-mat/9911107).

[14] T. Padmanabham, *Phys. Rep.* **188** (1990)285.

[15] S. A. Abel, J. L. F. Barbón, I. I. Kogan, E. Rabinovici, *JHEP* 9904 (1999) 015 (hep-th/9902058).

[16] A. A. Tseytlin, C. Vafa, *Nucl. Phys.* **B 372** (1992) 443.

[17] S. P. Patil, R. H. Brandenberger, (hep-th/0502069).

[18] D. Dryanov, O. Kounchev, "Polyharmonically exact formula of Euler-Maclaurin, multivariate Bernoulli functions, and Poisson type formula", *C. R. Acad. Sci. Paris, t.* **327***, issue 5, Série I-Math.* (1998) 515-520.